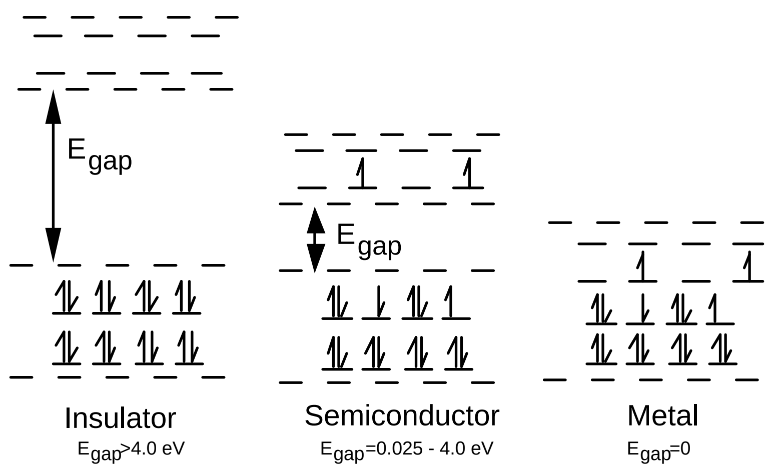
Chem 628
Lecture Notes
Passive Circuits: Resistance, Capacitance, Inductance

Our course in chemical instrumentation and electronics will begin with a quite overview of some things that you've probably seen before, but many of you will need a refresher. For some, it may be the first time you've encountered the material.

The main goals here are first, to give you a physical picture of what resistance, capacitance, and inductance arise. Then, we will want to analyze the relationship between voltage and current in perfect resistors, capacitors, and inductors. We'll then look at combinations, particularly combinations of resistors and capacitors, which are widely used as filters in circuits. If we have time, we'll also look briefly at some properties of inductors, which are of importance mainly in high-frequency circuits. Along the way we'll develop some mathematical tools to make the analysis of these circuits simpler, through the use of Kirchoff's Laws and, to some extent, through the use of complex notation.

Resistance:

Resistance is the tendency of a material to oppose the flow of *steady* current. In any material, the flow of electrical current requires partially-occupied electronic levels. Let's look at the three primary types of solid materials: insulators, metals, and semiconductors.

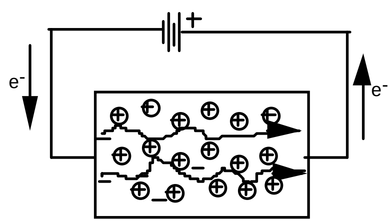


Insulators are materials in which there is a large gap in energy, E_{gap}, between the highest occupied electronic states and the lowest-lying unoccupied electronic states. In order for current to flow, an electron must be able to move from one atom to the next; however, because electrons are fermions, we know that no more than 2 electrons can occupy any given electronic energy level. In an insulator the lowest-lying energy level is completely filled, so that it can't hold any more electrons. In order for electrical current to flow in such a material, an electron must be excited from the highest-lying occupied energy state to the lowest unoccupied electronic energy level, which requires a great deal of energy. At room temperature, the average thermal energy of an electron is approximately equal to kT= 0.026 eV. Insulators have gaps which are many times larger than kT, usually greater than about 4 eV**. As a scientist, I think there are about 10 fundamental numbers that you have memorized in your head; the value of kT at room temperature is one of them.**

Semiconductors are materials in which there is again a gap between the highest occupied set of energy levels and the lowest unoccupied energy levels, but the gap is comparatively small. As a result, at room temperature some electrons will be thermally excited from the highest-occupied set of energy levels (referred to as the valence band) to the lowest-lying set of unoccupied energy levels (referred to as the conduction band). As we will discuss later, the electrical conductivity of semiconductors is strongly dependent on temperature and can also be strongly affected by optical illumination, making them useful for optical detectors. Semiconductors are typically defined as those materials which have gaps which are between about 0.1 eV and about 4 eV. (Silicon has a bandgap of 1.1 eV, or about 40 times larger than kT).

Metals are materials in which there is no gap at all between the occupied and unoccupied energy levels. The hybridization of s, p, and d-orbitals leads to a continuous distribution of energy levels (or an energy band) which is only partially occupied. As a result, electrons can easily be excited into unoccupied energy levels, leading to high electrical conductivity.

In all these materials, once an electron is excited into an empty energy level, it can propagate a relatively long distance. In a perfect crystalline lattice of a metal at zero Kelvin, the resistance would normally be quite small. Anything that impedes the flow of electrons increases resistance; since electrons aren't created or destroyed, these processes are referred to as "scattering" processes. As the temperature increases, several factors lead to increased resistance. First, the electrons can scatter from other electrons- this "electron-electron" scattering is usually negligible at room temperature. Secondly, the atomic nuclei begin to vibrate, leading to natural vibrational modes for the atomic nuclei ("phonons"); the electrons can sense the deviations of the nuclei from their ideal lattice positions and scatter off them via "electron-phonon" scattering. This is the dominant scattering mode in most metals at normal temperatures. Finally, if there are physical deviations from a perfect crystal lattice (point defects or chemical substitutions, as in an alloy) there will be "impurity" scattering.



Resistance

These factors lead to two general conclusions for electrical conductivity in metals:

1) Resistance increases as the temperature increases, because heat causes atoms in the crystal lattice to vibrate more (in the lingo of physics, we say there are more thermally-excited phonons for the electrons to scatter from).

2) Resistance increases as a material become more disordered. Consequently, alloys such as stainless steel (Fe, Ni, C) and BeCu have higher resistance than pure metals. In copper wires, th resistance is often controlled by oxygen impurities; it can be reduced by using "oxygen-free, high-conductivity" (OFHC) copper.

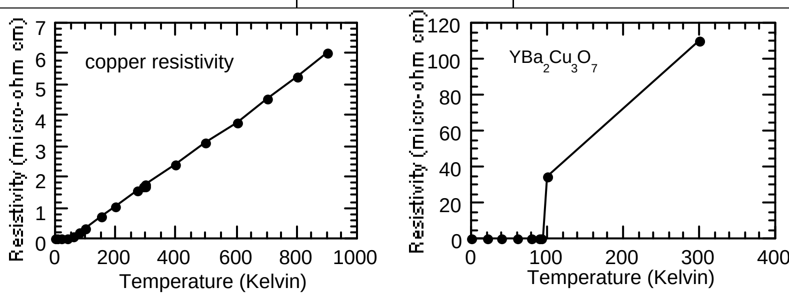
For semiconductors, the resistivity is a more complicated function of temperature. At low temperatures (such that kT is small compared with the bandgap E_{gap} of the semiconductor), an increase in the temperature increases the number of free electrons in the conduction band and decreases the resistance. At higher temperatures, scattering from phonons and defects again becomes important (as in a metal), and the resistance increases with increasing temperature.

An operational definition of resistance is Ohm's "Law", R=V/I, where V is the voltage across the material and I is the resulting current flow.

It is important to realize that Ohm's "Law" is not really a “law” at all, but rather a **definition** of what we call **resistance**. That resistance depends on a number of factors including the purity of the material, its crystalline form, the temperature, *and the applied voltage*. For metals the resistance is only weakly dependent on the applied voltage, so we can usually define a unique value of resistance that will hold for typical conditions. For semiconductors in complete devices, however, the current will often depend exponentially with voltage, so that there is not necessarily a well-defined value of "resistance".

The resistance of a meterial can be related to the resistivity ρ. For a material with cross-sectional area A and length l, the resistance and resistivity are related by the equation $R = \frac{\rho l}{A}$. Note that resistivity is usually expressed in units of ohm-cm, or sometimes, microohm-cm. Below are shown some values of resistivity for a few common metals:

Metal	Resistivity, ohm-centimeters	
Silver	1.6×10^{-6}	Best known conductor
Tantalum	1.6×10^{-6}	
Copper (OFHC)	1.7×10^{-6}	
Tungsten	5.6×10^{-6}	Light bulb filaments
Brass (Cu-Zn alloy)	7.0×10^{-6}	
Graphite	$65. \times 10^{-6}$	
Stainless steel	$74. \times 10^{-6}$	
Nichrome	$96. \times 10^{-6}$	Used as heater element in toasters
Silicon (doped)	0.01 - 10	
Silicon (pure)	100	
Silicon Carbide	150	
Pyrex Glass	10^{7}	
Fused Silica (quartz)	5×10^{9}	

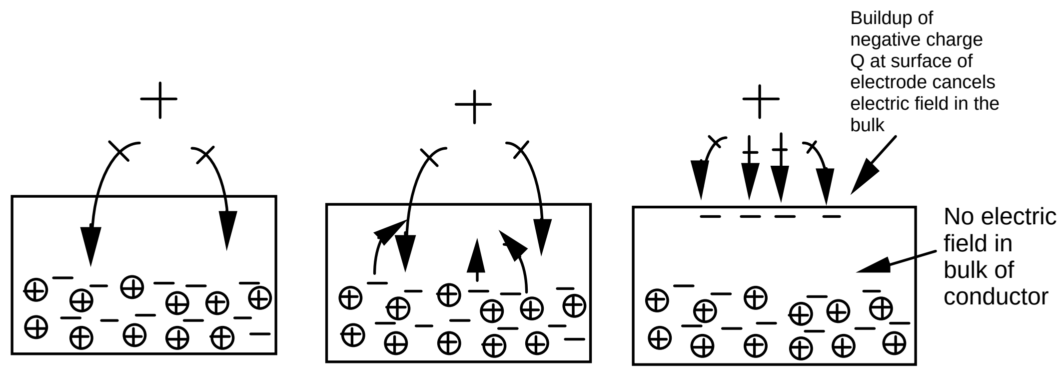


Conductivity vs. temperature for a typical metal (left) and a superconductor (right).

Capacitance:

From a fundamental standpoint, capacitance is the storage of charge. It is expressed mathematically as C=Q/V where Q is the stored charge (in Coulombs), V is the applied Voltage (V), and C is the capacitance . Capacitance has units of Coulombs/Volt, often referred to as a Farad (1 Farad = 1 Coulomb/Volt). A Farad is a huge capacitance, and more common values are microfarads, nanofarads, or picofarads (sometimes call micro-micro farads, or mmf, particularly in older literature). **Capacitance arises anytime we have two conductors separated by an insulator**.

Capacitance is easiest to understand in a the context of metallic electrodes. To understand capacitance, you must understand a basic law from electrostatics which states that an electric field cannot exist within a conductor without a resulting flow of current. For any static situtation (i.e., no corrent flow), this essentially means that no electric field can exist within a conductor. This is relatively easy to understand. Suppose that we have a good conductor, and somewhere outside of the conductor we bring a positive point charge.



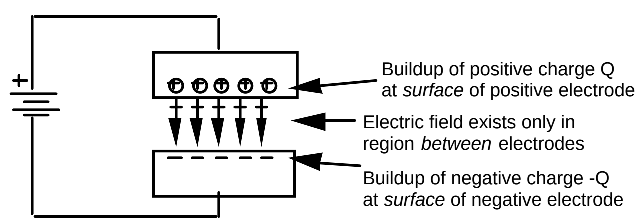
Electric field from + charge penetrates (momentarily) into bulk

Free electrons move in electric field, toward surface

Redistribution of electrons cancels electric field in bulk by buildup of surface charge Q

This point charge will tend to create an electric field within the conductor. However, if an electric field exists within a conductor, the "free" electrons will be acted upon by a force F=qE. For a positive point charge outside the metal, the electrons will tend to move toward the positive charge. This redistribution of charge within the metal tends to oppose the electric field induced by the positive charge, by creating an excess of electrons at the surface of the metal. The net result is that if we try to apply an electric field to a conductor, the free electrons within the conductor move to the surface in such a way as to cancel the electric field within the material.

Now consider the situation in which we have two metallic plates separated by a vacuum. If we apply a voltage between them, we create an electric field. The electrons in one plate will tend to move toward the surface to cancel the electric field within that electrode, and the electrons in the other will move away from the surface in order to cancel the electric field within that electrode. But there remains an electric field in the space between the metal plates. The buildup or deficiency of electrons near the surface of the conductor can be quantitatively described in terms of a buildup (or depletion) of charge Q. The capacitance C is then defined as C=Q/V; i.e., it is the ratio of stored charge to applied voltage. It can easily be shown from electrostatics that for a metal, the amount of stored charge is proportional to the applied voltage, so that C is a constant (nearly) independent of voltage, but dependent mainly on the physical arrangement of the metallic plates. As you might expect, the capacitance is proportional to the area of the plates and is inversely proportional to the separation between the plates. Taking into account all the proportionality constants and units conversions, the capacitance of a pair of parallel metal plates is given by: $C= e_0A/D$ where e_0 is a constant called the permittivity of free space, A is the area of the plates, and D is the distance between them.



If, instead of leaving a vacuum in the region between the plates, we instead put some kind of insulating material, the electric field between the plates will induce a polarization (distortion of the electron clouds) in this material. This *increases* the overall capacitance such that the new capacitance is given by $C = \varepsilon_0 \frac{K_D A}{D}$. where $e_0=8.854\times10^{-12}\ C^2N^{-1}\ m^{-2}$ (C=Coulomb, N = Newton, m= meter) is the permittivity of free space, and K_D is the dielectric constant (which is dimensionless).

Note that high dielectric constants lead to high capacitance, and vice versa. Below are shown some dielectric constants for important materials:

Material	Dielectric Constant
Vacuum:	1.0000 (by definition)
Teflon	2.1
Polystyrene	3.2
Polyester	3.5
Mica	5.4
Aluminum Oxide	8.4
Tantalum Oxide	27.6
Hafnium Oxide	25
Barium Titanate	200

Practical Capacitors:

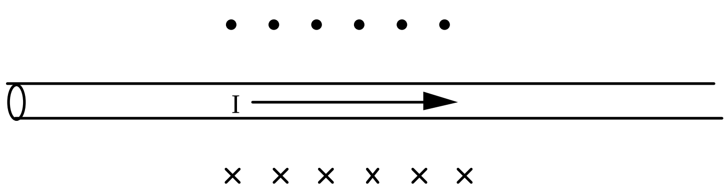
Capacitors used for electronics applications should have several properties:

1) Capacitance should be independent of voltage, temperature and humidity
2) There should be no inductance or resistance for an ideal capacitor
3) The capacitor should have a reasonable physical size

In practice, it is difficult to meet all of these conditions simultaneously. As a result, there are several kinds of capacitors available, each with its own advantages and disadvantages. One of the primary difficulties is that because of imperfections in the dielectrics used in real capacitors, their "resistance" is not infinite (i.e., when a constant voltage is applied to the capacitor, some "leakage" current will flow). Whether leakage is important or not depends on the application. A good table of the advantages and disadvantages of each type of capacitor is given in Horowitz and Hill "The Art of Electronics", page 22.

Inductance

Inductance results from the fact that a flow of current produces a magnetic field, according to the "right-hand rule". However, a changing magnetic field also tends to induce a voltage(or current) in a wire according to Faraday's Law: $V=-d\Phi/dt$, where Φ is the magnetic flux. (This is the basic principle of magnetic induction). The net result is that whenever we try to create a flow of current in a wire, the resultant magnetic field will couple back and will tend to oppose the **change** in current that we're trying to make. Note that the effect depends on *the rate of change* of current, not on the value of the steady current itself.



Magnetic field produced by a current according to the Right-Hand Rule

The operational definition of inductance is: $L = \frac{V}{dI/dt}$.

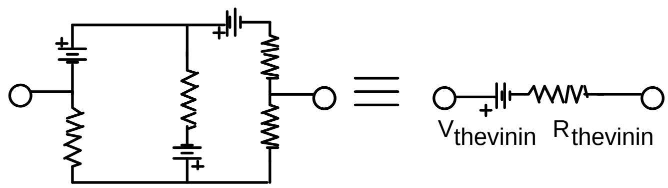
The unit of inductance, Volts-seconds/Amp, is called a "Henry". Typical inductances are in the microhenry to millihenry range.

Every wire in an electric circuit has some associated inductance. The electric device that we call an "inductor" is usually a coil of wire, often wrapped around some ferromagnetic material such as iron. Note that for a circuit with a large inductance, it is difficult to change the current rapidly (i.e., dI/dt will be small).

Important!!
There is no such thing as a "perfect" resistor, capacitor, or inductor. All electrical components will have some resistance, capacitance, and inductance. The absence of "perfect" devices often limits the performance of circuits, especially at high frequencies.

Thevinin's Theorem:

Thevenin's theorem says that any circuit having only two terminals and consisting of any number of voltage sources and resistors can be replaced by a single equivalent voltage source in series with a single resistor. These are sometimes called the "Thevenin equivalent voltage" and the "Thevinin equivalent resistance.



We will be using Thevinin's Theorem extensively throughout the semester.

Kirchoff's Laws:

Kirchoff's Laws are a general way of keeping track of voltages and currents in a circuits containing more than one resistor, capacitor, or inductor. Kirchoff's Laws state that:

1) At any *point* in a circuit, $\sum I = 0$

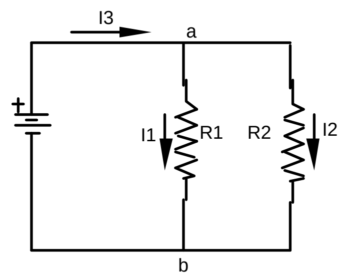
2) Around any *closed loop*, $\sum Voltage \cdot drops + \sum V_{sources} = 0$

Kirchoff's First Law is another way of expressing **conservation of charge**. It simply says that at any point in a circuit, we can't accumulate or deplete charge. (If we try to apply this to a point right on the surface of a capacitor plate we run into trouble, but otherwise it's fine..)

Kirchoff's Second Law is another way of expressing **energy conservation**. It says that the total energy change experienced by an electron flowing in a complete loop must be zero. (Or equivalently, it says that the electrostatic potential is a thermodynamic state function, and we can't violate the second law of thermodynamics!). For a circuit consisting only of batteries and resistors, the "voltage drops" are given by Ohm's Law: V=IR.

In using Kirchoff's Laws, the biggest challenges are usually remembering that currents and voltages are **signed** quantities. As a simple illustration of Kirchoff's Laws, let's calculate the equivalent resistance of two resistors in parallel.

The first step is to assign currents in each leg of the circuit, with a direction, as I_1, I_2, etc. An important point here is that in making these initial assignments**, we do not need to know the actual direction of current flow**, but are simply making assignments of what we consider "positive" for use in solving the mathematical equations. If the final currents that result from solution of the equations are positive, it means that our initial assignment has the correct direction. If the final currents that result from solution of the equations are negative, it means that our initial assignments have the wrong direction (the magnitude of the quantities will still be correct). Thus, the choice of directions is completely arbitrary. We will get the correct answer in any case. Let's choose a really simple example first.



We'll pick the point labeled "a". Here, Kirchoff's first law states that $I_3 - I_2 - I_1 = 0$. By convention, we use positive signs for currents that we have assigned flowing toward the point, and negative for currents that are flowing away.

For Kirchoff's Second Law, we'll pick the loop that includes the battery and one resistor. Note that we must specify an (arbitrary, but consistent) direction for the loop. In calculating voltage drops around a loop, the drop is positive if the assigned current direction and the loop direction are the same; otherwise the voltage drop is negative. Likewise, the sign of a voltage source (battery) is positive if the current loop passes from the positive battery pole, through the circuit, and returns to the negative pole of the battery. In the case above, we have $V_{battery}$ = Va-Vb = Vab.

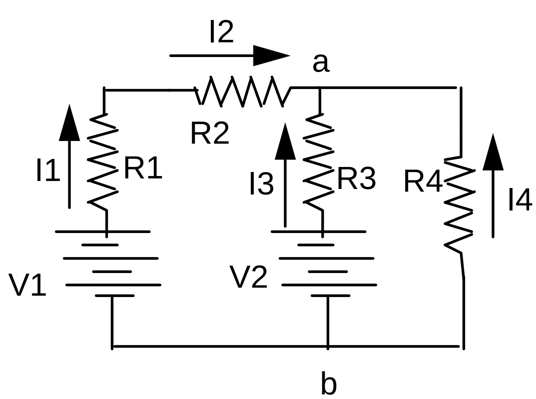
Now we can solve the equation. We know that $I_1 = V_{ab}/R_1$ and $I_2 = V_{ab}/R_2$, from Ohm's Law.

Then, $I_3 = I_1 + I_2 = V_{ab}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ We can rewrite this in terms of an "equivalent" parallel resistance R_{eq} as $I_3 = \frac{V_{ab}}{R_{eq}} = \frac{V_{battery}}{R_{eq}}$ I where $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R2}$, or equivalently, $R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$.

Thus, we've used Kirchoff's Laws to derive the well-known expression for the Thevinin equivalent resistance of two parallel resistances.

Similarly, we could find the expression for the Thevinin equivalent resistance of two series resistances:

Let's consider a harder case.



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At point "a" we have $I_2+I_3+I_4=0$ (1)

At point "b" we have $-I_1-I_3-I_4=0$ (2)

For the loop labeled "A" we have

$V_1-V_2=I_1R_1+I_2R_2-I_3R_3$. (3)

For the loop labeled "B" we have

$V_2=I_3R3-I_4R_4$. (4)

We now have four equations in four unknowns and can solve; this can be done in a variety of ways. For more complicated circuits, matrix algebra is the best. For small systems, we can solve by simple elimination.

Subtracting equations 1 and 2 gives: $I_1=I_2$ (5)

This gets us down to three equations (6-8):

Substitution $I_1=I_2$ into equation (1) gives $I_1+I_3+I_4=0$ (6)
Substituting $I_1=I_2$ into equation (3) gives $V_1-V_2=I_1(R_1+R_2)-I_3R_3$ (7)
and we still have equation (3) unchanged: $V_2=I_3R_3-I_4R_4$ (8)

Solving equation (8) for I4 gives $I_4=(I_3R_3-V_2)/R_4$, and substitution of this into equations (6) and (7) gives:

$I_1+I_3 + (I_3R_3 - V_2)/R_4 =0$ (9)
$V_1-V_2=I_1(R_1+R_2)-I_3R_3$ (10) (this is the same as equation (8)).

Now we're down to 2 equations (9 and 10). Solving equation (10) for I_3 gives $I_3=(I_1(R_1+R_2)-V_1+V_2)/R_3$, and substituting this into equation (9) gives

Now it's just rearranging:

And finally (assuming that I haven't made any mistakes!!!)

Now you can back-substitute to find the other currents and voltages. We won't do this here because I'm getting tired of using the equation formatter, and by now you should have the general idea.

Two more tips on using Kirchoff's Laws:

1) Remember that you only need as many (independent) equations as unknowns. In the above example we had 4 unknown currents, and so we needed only four equations. We could have generated more equations by choosing a different loop (say, but taking the large loop through V1,R1,R2, and R4). However, those equations would not have been independent equations - they would have been linear combinations of the four equations which we used. So, don't go overboard - think about how many equations you need, and use only that many. Otherwise you'll run into trouble.

2) Kirchoff's are in fact general, and can include resistors, capacitors, inductors, etc, provided that you properly express the voltage across the capacitor or inductor. Since for capacitors and inductors the current isn't just proportional to the voltage, this usually leads to a differential equation, rather than a simple linear algebra problem.

Voltage Dividers

Above, we saw the basic formulas for the equivalent resistance of two series resistors and two parallel resistors. These can be generalized to "N" resistors as follows:

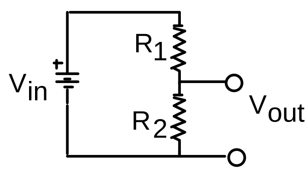
Parallel Resistors:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots + \frac{1}{R_N}$$

Series Resistors:

$$R_{eq} = R_1 + R_2 + R_3 + \cdots + R_N$$

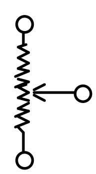
Let's got back to the case of two series resistors, and measure the voltage across R_2. We'll call this V_{out}.:



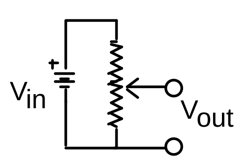
The current in both resistors is the same. The Thevinin equivalent resistance is $R_{eq} = R_1+R_2$, so that $I = \frac{V}{R_1 + R_2}$, and the voltage drop across resistor R_2 is just IR_2. Then,

$$V_{out} = \frac{V_{in} R_2}{R_1 + R_2} = \frac{R_2}{R_1 + R_2} V_{in}$$

This is known as the VOLTAGE DIVIDER EQUATION. It's one of the equations that you should memorize. The voltage divider is used in many applications. By varying the values of the resistors, the output voltage can be changed. One common application of a voltage divider is as a potentiometer. A potentiometer is a three-terminal resistor, in which there is a movable "slider" contact. The total resistance between two terminals is always constant, but turning a knob (or sliding a level) the position of the "slider" can be changed, thereby changing the ratio $R_2/(R_1+R_2)$.



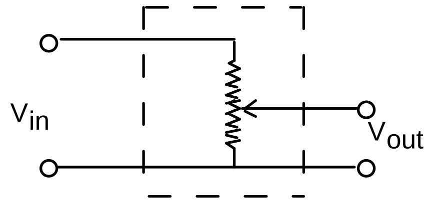
Potentiometer



Potentiometer as
Variable Voltage Divider

The Voltage Divider is widely used as a way of conveniently adjusting an output voltage. In other applications, only two terminal of the potentiometer will be used, in which case it is acting simply as a variable resistor.

In our study of electronics, we will often try to think of complicated electronic circuits as being comprised of a number of smaller "modules".   A voltage divider is a good example of a simple module which has an input voltage and an output voltage.



Potentiometer as
An Instrument Module

When we talk about instruments being composed of modules, we will generally be interested in describing the operation of each module using some sort of common terminology. The terms that we will use in describing the operation of any module are:

1) **Transfer Function** (The ratio of output to input, including phase information)

2) **Input Impedance**

3) **Output Impedance.**