

Chem 628

Voltage Dividers, Transfer Functions, and Input and Output Impedances

We'll use the idea of a transfer function quite a bit. Because usually in electronic circuits we are interested in measuring voltages (rather than currents), we'll define the transfer function in terms of voltages: $T \equiv \frac{\overline{V_{out}}}{\overline{V_{in}}}$, where the vertical bars denote root-mean-square (RMS) values (irrelevant for DC circuits, but important for AC circuits). The RMS value of an AC circuit is defined as:

$V_{RMS} = \sqrt{\langle V^2 \rangle}$. That is, it is the square **root** of the **mean** (average) value of the **square** of V. For a periodic signal with period T this is given by:

$$V_{RMS} = \sqrt{\frac{1}{T} \int_0^T (V(t))^2 dt}$$

For the voltage divider circuit described earlier, we saw that the input and output voltages were related by: $V_{out} = \frac{R_2}{R_1 + R_2} V_{in}$, so that the transfer function is simply $T = \frac{R_2}{R_1 + R_2}$. Note that the transfer function is a ratio of two numbers, and so it is a dimensionless (pure) number. When we get to AC circuits, we'll see that the transfer function of circuits that include resistors and capacitors will be a function of frequency. For reasons that will become clear later, it is common to plot transfer functions on a logarithmic scale, in units of "decibels".

Most people find decibels confusing at first. The "bel" was originally invented as a unit of acoustic power back in the early days of the telephone. The "bel" part is named after Alexander Graham Bell, who invented the telephone. Because the "bel" is a unit of power (not voltage), there's an extra conversion factor that comes in. You will (hopefully) remember that power is defined as current x voltage ($P=V \times I$), which for a resistive circuit becomes $P=V^2/R$. For a purely

resistive circuit, $1\text{Bel} = \log_{10} \left(\frac{P_{out}^{rms}}{P_{in}^{rms}} \right) = \log_{10} \left\{ \left(\frac{V_{out}^{rms}}{V_{in}^{rms}} \right)^2 \right\} = 2 \log_{10} \left(\frac{V_{out}^{rms}}{V_{in}^{rms}} \right)$. , where "rms" refers to

the "root-mean-square" value (more about this when we do AC circuits). A "decibel" is one-tenth of a bel, so that $T(\text{decibels}) = 20 \log_{10} \left(\frac{V_{out}^{rms}}{V_{in}^{rms}} \right)$. This, a transfer function of 0 decibels means that $T=1$

(i.e., the output voltage is equal to the input voltage). A transfer function of -20 decibels (or -20 dB) means that the output voltage is 0.1 of the input voltage, and a transfer function of -40 decibels (-40 dB) means that the output voltage is 0.01 of the input voltage (remember, it's a logarithmic scale!!!).

The other important property of any instrument module is the phase shift encountered by an AC signal passing through. If an input signal is represented as $V_{in} = A \cos \omega t$ and the output is represented as $V_{out}(t) = B \cos \omega t + C \sin \omega t$, then the transfer function is:

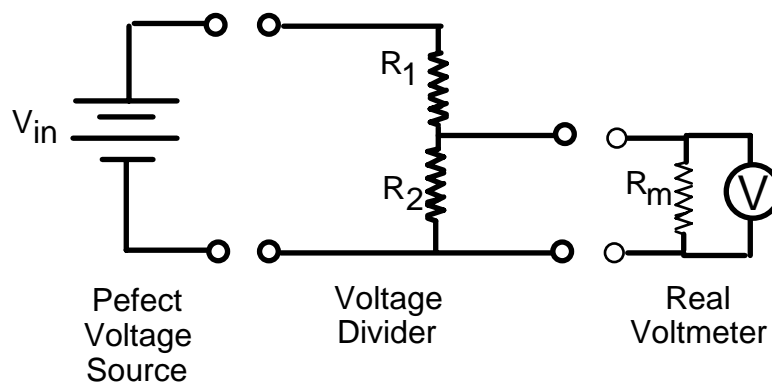
$$T = \frac{\sqrt{B^2 + C^2}}{A}, \text{ and the phase shift is given as: } \phi = -\arctan\left(\frac{C}{B}\right)$$

We will cover phase shifts in more detail when we talk about AC circuits.

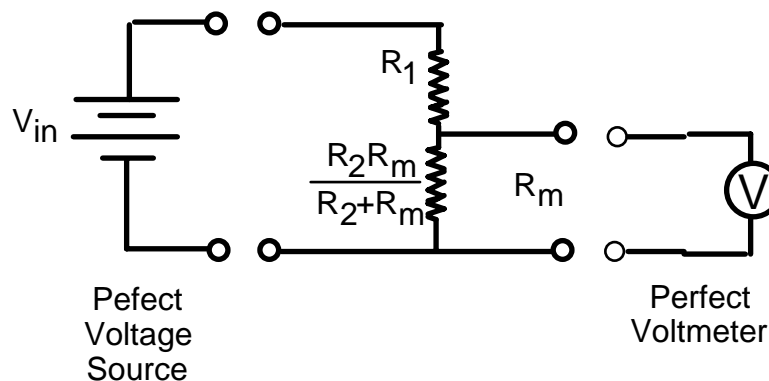
Input and output impedance:

To look at input and output impedance, it's useful to consider a simple circuit in which we look upon a voltage divider as a one module, and look the effect of hooking up a second module to the output of the first. The second module will be a model for a voltmeter. A "perfect" voltmeter measures voltage without drawing any current. A real voltmeter always draws some current, and so it can be modelled as an "ideal" voltmeter in parallel with a resistor as shown below.

The basic idea in understanding input and output impedances is based on a model in which the output of any stage can be modeled as a battery in series with a resistor (the output impedance), and the input of any stage can be modelled as a resistor (the input impedance) in parallel with a "perfect" voltmeter. Input and Output Impedances are important because they reflect how accurately voltages will be measured, and also how much power is transferred from one module to another.



We're interested in determining what voltage is actually measured by the "real" voltmeter V . Looking at the circuit in a bit more detail, you'll see that it can be redrawn as shown below.



Now you can probably see that what the voltmeter will actually measure is given

$$\text{by: } V_{\text{measured}} = \frac{\frac{R_2 R_m}{R_2 + R_m}}{R_1 + \frac{R_2 R_m}{R_2 + R_m}} V_{in} = \frac{R_2 R_m}{R_1 (R_2 + R_m) + R_2 R_m} V_{in} = \frac{R_2 R_m}{R_1 R_2 + R_1 R_m + R_2 R_m} V_{in} . \text{ This is}$$

obviously more complicated than what we had originally for a simple voltage divider. Let's look at some limits. First, let's look at what happens of $V_m \gg R_1$ and $R_m \gg R_2$. Then, the

denominator of the equation (in its last form) simplifies to $R_1 R_m + R_2 R_m$, and the overall equation simplifies to: $V_{measured} = \frac{R_2 R_m}{R_1 R_m + R_2 R_m} V_{in} = \frac{R_2}{R_1 + R_2} V_{in}$, and we have exactly what we had originally expected. In this example, we see that the fact that a "real" voltmeter does not have infinite resistance (and hence, draws some current from the circuit under measurement) introduces some error. The concept of "input impedance" and "output impedance" is a way of quantifying this. As we'll see, "impedance" is simply a generalized form of resistance which is applicable to circuits containing resistors, capacitors, and inductors. For now, we'll consider "impedance" to be synonymous with "resistance", until we start dealing with AC circuits.

When constructing an instrument by cascading individual modules into a complete system, we can think of the input and output impedances of each individual module separately. That greatly simplifies the overall design of the system. Also, remember that we are usually dealing with signals which have a "direction"- that is, the signal starts at one place and might be amplified, filtered, etc., in order to come up with some final, processed signal. The important concept here is that each module has a well-defined "input" and "output" which are determined by the direction which our "signals" are flowing. For something like a voltage divider, it's not immediately obvious that either end has to be an "input" and the other end an "output". For most electronic circuits that we'll be talking about, however, input and output cannot be interchanged.

The input impedance of a module is a measure of how much current the module draws from the preceding module. From our example above, we saw that if the resistance (impedance) of our "real" voltmeter, R_m , was large compared with R_1 and R_2 , then the "real" voltmeter would accurately measure what we expected. If, however, R_m was smaller, then our measurements would be in significant error.

Likewise, the output impedance of a module is a quantity which tells us how small a resistor could be connected across the output before the output voltage would significantly deviate from its "ideal" value. In general, definitions of "input" and "output" impedance make the assumption that, based on the Thevenin theorem, the output of any circuit can be equated to a single voltage source in series with a single resistor.

What is the Thevenin Voltage and Thevenin Resistance?

The Thevenin Voltage is defined as the output voltage of a circuit under "no-load" conditions - that is, under conditions where the measuring voltmeter draws no current, or equivalently, has infinite resistance: $R_m = \infty$. For the simple voltage divider, we already calculated the

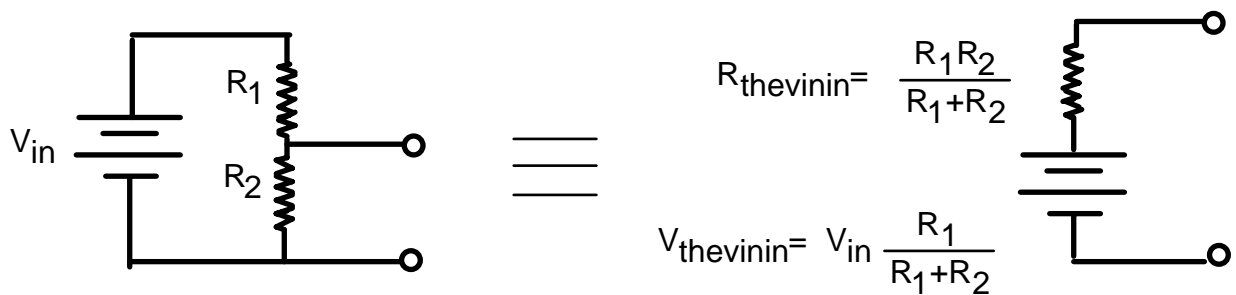
Thevenin voltage as $V_{thevenin} = \frac{R_2}{R_1 + R_2} V_{in}$.

The Thevenin Resistance is defined in a little more complicated way. If we short-circuit the output of the voltage divider, we can measure the current flow, $I_{short-circuit}$. The Thevenin resistance $R_{thevenin}$ is defined as $R_{thevenin} = V_{thevenin} / I_{short-circuit}$.

For the voltage divider, the short-circuit current will be $I_{short-circuit} = V_{in} / R_1$. Therefore, the Thevenin Resistance for the Voltage Divider is:

$$R_{thevenin} = \frac{V_{thevenin}}{I_{short-circuit}} = \frac{\frac{R_2}{R_1 + R_2} V_{in}}{\frac{V_{in}}{R_1}} = \frac{R_1 R_2}{R_1 + R_2}$$

Note that this is just the parallel resistance of R1 and R2 !! Another way of looking at the Thevinin resistance is that it is essentially the resistance that would be measured if you connected an ommeter across the output terminals of the "module" (here, a voltage divider). Thus, the Thevinin model for the voltage divider circuit (including the battery) is:



Voltage Divider with Battery

Thevinin Equivalent

Going back to our earlier circuit which included a "real" voltmeter, we can calculate the overall output voltage measured by the system using Rthevinin, Vthevinin, and Rm. Now, Rthevinin and Rm form a simple voltage divider, and

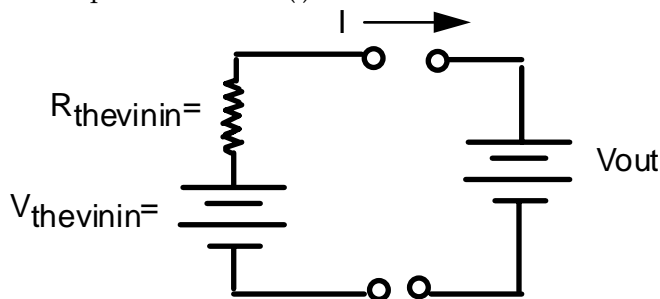
$$V_{measured} = \frac{R_m}{R_{thevinin} + R_m} V_{thevinin} = \frac{R_m}{\frac{R_1 R_2}{R_1 + R_2} + R_m} \frac{R_1}{R_1 + R_2} V_{in} = \frac{R_m R_1}{R_1 R_2 + R_m (R_1 + R_2)} V_{in} = \frac{R_m R_1}{R_1 R_2 + R_1 R_m + R_2 R_m} V_{in}$$

Note that this is exactly what we had before!!!! The Thevinin model allows us to simplify the output of the circuit containing a battery and two resistors into a simpler circuit consisting of a single battery and a single resistor!! (You might argue that it's a lot of work, but of course, it's more general than this!!)

The current through the resistor Rm will be $I_m = \frac{V_m}{R_m} = \frac{R_1 V_{in}}{R_1 R_2 + R_1 R_m + R_2 R_m}$

The output impedance of a circuit is defined a little differently: It's defined as $Z_{out} = \frac{\Delta V_{out}}{\Delta I_{out}}$.

where Vout and Iout are changes in the current and voltage produced by applying a voltage to the output of the circuit(!).



For purely resistive circuits, the output impedance $Z_{out} = R_{thevinin}$.

This definition may seem strange at first, but it is used because for non-resistive circuits, the output voltage and current are not necessarily related by Ohm's Law.

In order for an instrument "module" to accurately measure the voltage produced by a previous module, the requirement is that

