

Chem 628: Fall 2005  
Lecture Notes  
Analysis of RLC Circuits

For even relatively simple circuits such as low-pass and high-pass filters, most problems will typically involve solving a differential equation. However, even relatively complicated circuits consisting of resistors, capacitors, inductors, and voltage sources can be analyzed much more easily by using complex notation. This method will work for all RLC circuits.

To do this, we first define a complex voltage  $\mathbf{V}$ . (Bold-face letters will denote complex quantities). The real (measurable) voltage is then  $V_{\text{measured}} = \text{Re}(\mathbf{V})$ . Likewise, we define a complex current  $\mathbf{I}$ . The real (measurable) current is then  $I_{\text{measured}} = \text{Re}(\mathbf{I})$ . Then, we define the complex impedance as:

$$\mathbf{Z} \equiv \frac{\mathbf{V}}{\mathbf{I}}$$

**Why does this method work?**

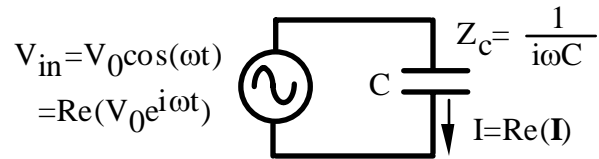
Complex notation allows us to include the fact that for an AC input signal, the output signal will generally be shifted in phase with respect to the input signal, but will still be proportional to the input signal- i.e., the response is still linear.

Note that current and voltage are directly-measurable quantities, and so must be REAL numbers. By representing the impedance as a complex number, we allow devices like inductors and capacitors to change the *phase* of the output signal relative to the input signal, in addition to changing the amplitude.

For simplicity, we usually assume that the input signal in complex notation is  $\mathbf{V} = V_0 e^{i\omega t} = V_0(\cos(\omega t) + i \sin(\omega t))$ . Then,  $\text{Re}(\mathbf{V}) = V_0 \cos(\omega t)$ . This is essentially the same as what we assumed in our earlier analysis of the low-pass filter, in which we assumed that the input signal was of the form  $V_{\text{in}} = V_0 \cos(\omega t)$  and then solved a first-order differential equation to get the output signal.

**Example 1:**

Let's look at a simple example first.



Here, from the definition of capacitance we have  $Q = CV$ . Differentiating, we get  $\mathbf{I} = \frac{dQ}{dt} = C \frac{dV}{dt}$ . If  $\mathbf{V} = V_0 e^{i\omega t}$ , then  $\frac{dV}{dt} = i\omega V_0 e^{i\omega t}$ , so that  $\mathbf{I} = i\omega C V_0 e^{i\omega t} = i\omega C \mathbf{V}$ .

Then,  $\mathbf{Z} \equiv \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{i\omega C}$  **This is defined as the complex impedance of a capacitor.**

Since  $\mathbf{I} = i\omega V_0 e^{i\omega t}$ ,

$$I_{measured} = \text{Re}(\mathbf{I}) = \text{Re}(i\omega V_0 \cos(\omega t) - \omega V_0 \sin(\omega t)) = -\omega V_0 \sin(\omega t)$$

This is identical to what we found earlier as the solution to the differential equation.

## Inductors

As we saw before, the inductance is defined by  $L \equiv \frac{V}{dI/dt}$ , so that  $dI/dt = \frac{V}{L}$ . If we

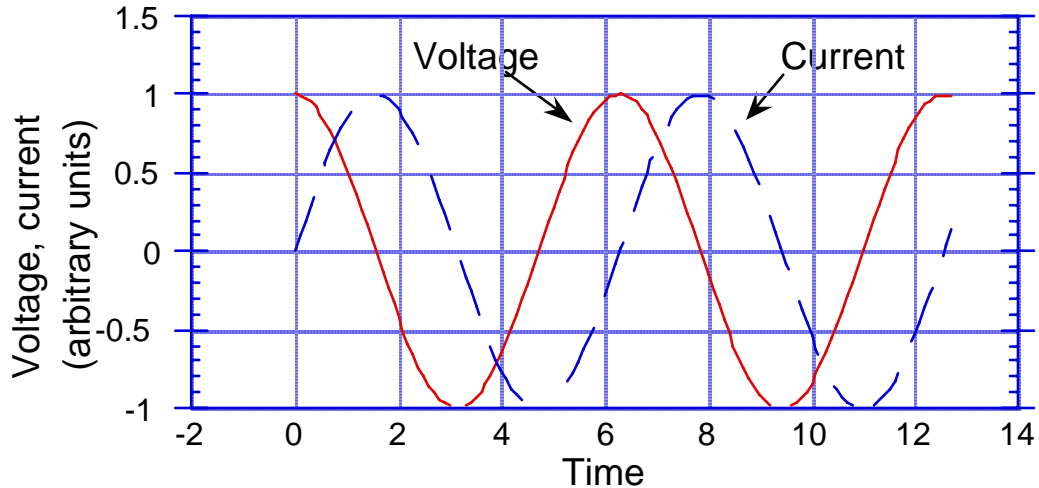
assume that the input voltage has the form  $V_{applied} = V_0 \cos(\omega t)$ , then

$dI/dt = \frac{V_0}{L} \cos(\omega t)$ . Then,  $I(t) = \frac{V_0}{L} \int \cos(\omega t) dt$ . The solution to this

is  $I(t) = \frac{V_0}{L\omega} \sin(\omega t) + C$ , where  $C$  is any constant. We can often ignore this constant value (which is essentially a result of the boundary conditions), and write

$$I(t) = \frac{V_0}{\omega L} \sin(\omega t) = \frac{V_0}{\omega L} \cos(\omega t - 90^\circ).$$

If we plot the current through the inductor as a function of time, it looks like:



Here, the current reaches a maximum value  $\pi/2$  (or 90 degrees) after the voltage does. Thus, the current through an inductor is said to lag the voltage by 90 degrees.

If we use complex notation, then we can again write the applied voltage as  $V_{in} = V_0 e^{i\omega t}$ . We then have  $I = \frac{1}{L} \int V(t) dt = \frac{1}{L} \int V_0 e^{i\omega t} dt = \frac{1}{i\omega L} V_0 e^{i\omega t} = \frac{1}{i\omega L} V(t)$

Using the definition of complex impedance gives:

$Z \equiv \frac{V}{I} = \frac{V}{\frac{V}{i\omega L}} = i\omega L$ . This then becomes the **definition** of the complex impedance

of an inductor.

So, we have:

$$Z_{resistor} = R$$

$$Z_{capacitor} = \frac{1}{i\omega C} = \frac{-i}{\omega C}$$

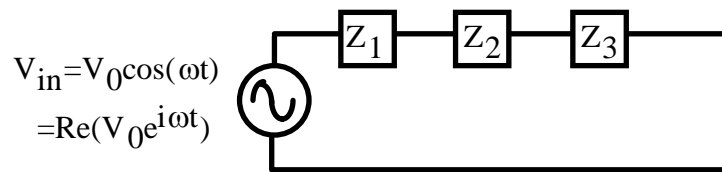
$$Z_{inductor} = i\omega L$$

### Combinations of Resistors, Inductor, and Capacitors:

The real power of the complex notation method comes in when we deal with even more complicated situations.

Let's look at some general rules:

#### Complex impedances in series:

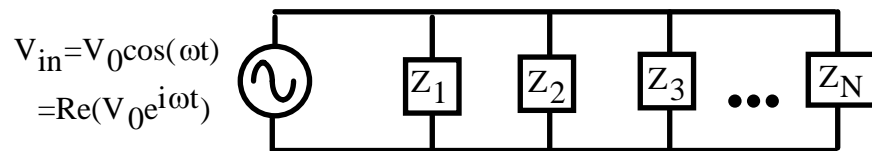


Kirchoff's Laws give:

$$I = \frac{V_1}{Z_1} = \frac{V_2}{Z_2} = \frac{V_3}{Z_3}, \text{ and } V = V_1 + V_2 + V_3$$

$$V = IZ_1 + IZ_2 + IZ_3 = I(Z_1 + Z_2 + Z_3) = IZ_{effective}, \text{ where } Z_{effective} = Z_1 + Z_2 + Z_3$$

Complex Impedances in parallel:

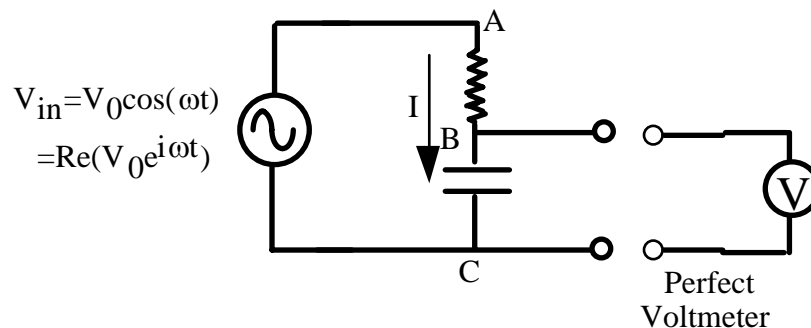


$$I = I_1 + I_2 + I_3 + \dots = \frac{V}{Z_1} + \frac{V}{Z_2} + \frac{V}{Z_3} + \dots = V \left( \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right) = \frac{V}{Z_{effective}} \text{ where}$$

$$\frac{1}{Z_{effective}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \dots$$

**Thus, the rules for adding complex impedances in series and parallel are identical to the rules for adding resistances.**

Example 2: Low-pass filter



In this case we know from Kichoff's Laws that  $\mathbf{V}=\mathbf{V}_{AB}+\mathbf{V}_{BC}$ . and  $I_{AB}=I_{BC}$  Using ohm's law (for the resistor) and definition of complex impedance for a capacitor which we just determined above gives:

$$\mathbf{V}_{AB}=\mathbf{I}R \text{ and } \mathbf{V}_{BC}=\mathbf{I}Z_{\text{capacitor}} = \mathbf{I}\left(\frac{1}{i\omega C}\right)$$

$$\mathbf{V} = \mathbf{I}R + \mathbf{I}\left(\frac{1}{i\omega C}\right) = \mathbf{I}\left(R + \frac{1}{i\omega C}\right) , \text{ or } \mathbf{I} = \frac{\mathbf{V}}{R + \frac{1}{i\omega C}} . \text{ Substituting this expression for I}$$

into the above expression for  $\mathbf{V}_{BC}$  gives:

$$\mathbf{V}_{BC} = \mathbf{I}Z_{\text{capacitor}} = \frac{\mathbf{V}}{R + \frac{1}{i\omega C}} \left(\frac{1}{i\omega C}\right) = \frac{\mathbf{V}}{1 + i\omega RC} = \mathbf{V} \left(\frac{1 - i\omega RC}{1 + (\omega RC)^2}\right) .$$

Now, we remember that  $V_{BC(\text{measured})} = \text{Re}(\mathbf{V}_{BC})$ . However, since both  $\mathbf{V}$  and  $\left(\frac{1 - i\omega RC}{1 + (\omega RC)^2}\right)$  are complex, we need to be careful:

$$V_{BC(\text{measured})} = \text{Re}(\mathbf{V}_{BC}) = \frac{1}{1 + (\omega RC)^2} \text{Re}(\mathbf{V}) + \frac{\omega RC}{1 + (\omega RC)^2} \text{Im}(\mathbf{V}) .$$

But,

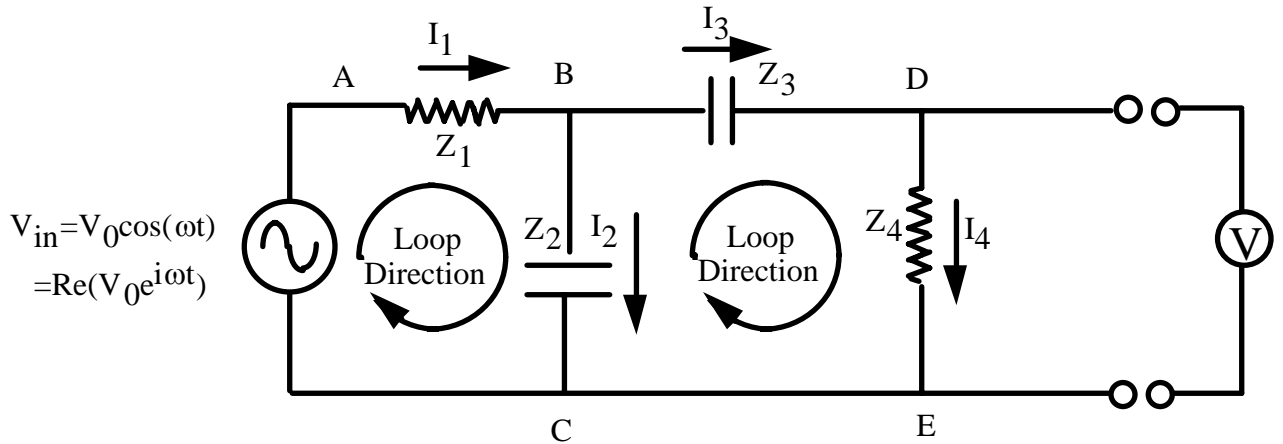
$\mathbf{V}=\mathbf{V}_0e^{i\omega t}$ , so  $\text{Re}(\mathbf{V})=V_0\cos(\omega t)$  and  $\text{Im}(\mathbf{V})=V_0\sin(\omega t)$ , so that

$$V_{BC(\text{measured})} = \frac{1}{1 + (\omega RC)^2} \cos(\omega t) + \frac{\omega RC}{1 + (\omega RC)^2} \sin(\omega t)$$

This is exactly the same solution as we found earlier. In this case, the complex notation saved us from having to solve a differential equation.

### **A more complicated example: The RC Bandpass Filter:**

This is an example where if you didn't have complex notation available you'd probably just throw up your hands and give up. With complex notation it's straightforward, although possibly tedious.



Kirchoff's Current laws give:  $I_1 - I_2 - I_3 = 0$  and  $I_3 = I_4$ .

Kirchoff's Voltage Laws give:

$$V = V_{AB} + V_{BC} = I_1 Z_1 + I_2 Z_2$$

$$-V_{BC} + V_{BD} + V_{DE} = 0 = -I_2 Z_2 + I_3 Z_3 + I_4 Z_4$$

We thus have four equations in four unknowns ( $I_1 \dots I_4$ ) If we want to find the output voltage, we need to find  $I_4 (= I_3)$ .

The four equations above can be easily simplified to three equations in three unknowns:

$$0 = I_1 - I_2 - I_3$$

$$V = I_1 Z_1 + I_2 Z_2$$

$$0 = -I_2 Z_2 + I_3 (Z_3 + Z_4)$$

or, in matrix notation,

$$\begin{pmatrix} 1 & -1 & -1 \\ Z_1 & Z_2 & 0 \\ 0 & -Z_2 & Z_3 + Z_4 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0 \\ V \\ 0 \end{pmatrix}$$

The solution to this equation is:

$$I_4 = I_3 = \frac{V Z_2}{Z_1 \left[ Z_2 + (Z_3 + Z_4) \left( \frac{Z_2}{Z_1} + 1 \right) \right]}$$

Substituting in  $Z_1 = R_1$ ,  $Z_2 = \frac{1}{i\omega C_2}$ ,  $Z_3 = \frac{1}{i\omega C_3}$ ,  $Z_4 = R_4$  gives:

$$\mathbf{I}_4 = \frac{\mathbf{V}}{i\omega R_1 C_2} \left[ \frac{1}{\frac{1}{i\omega C_2} + \left( \frac{1}{i\omega C_3} + R_4 \right) \left( \frac{1}{i\omega R_1 C_2} + 1 \right)} \right]$$

$$\mathbf{I}_4 = \frac{\mathbf{V}}{\frac{i\omega R_1 C_2}{i\omega C_2} + i\omega R_1 C_2 \left( \frac{-1}{\omega^2 R_1 C_2 C_3} - \frac{iR_4}{\omega R_1 C_2} - \frac{i}{\omega C_3} + R_4 \right)}$$

$$\mathbf{I}_4 = \frac{\mathbf{V}}{R_1 + \frac{-i}{\omega C_3} + R_4 + R_1 \left( \frac{C_2}{C_3} \right) + i\omega R_1 C_2 R_4}$$

$$\mathbf{I}_4 = \frac{V_0 (\cos(\omega t) + i \sin(\omega t))}{R_1 + R_4 + R_1 \left( \frac{C_2}{C_3} \right) + i \left( \frac{-1}{\omega C_3} + \omega R_1 C_2 R_4 \right)}$$

$$\mathbf{I}_4 = \frac{V_0 (\cos(\omega t) + i \sin(\omega t)) \left[ \left( R_1 + R_4 + R_1 \left( \frac{C_2}{C_3} \right) \right) - i \left( \frac{-1}{\omega C_3} + \omega R_1 C_2 R_4 \right) \right]}{\left( R_1 + R_4 + R_1 \left( \frac{C_2}{C_3} \right) \right)^2 + \left( \frac{-1}{\omega C_3} + \omega R_1 C_2 R_4 \right)^2}$$

Then, the final voltage drop measured across  $R_4$  is given by:

$$\mathbf{V}_{DE} = \mathbf{I}_4 R_4 = \frac{V_0 R_4 (\cos(\omega t) + i \sin(\omega t)) \left[ \left( R_1 + R_4 + R_1 \left( \frac{C_2}{C_3} \right) \right) - i \left( \frac{-1}{\omega C_3} + \omega R_1 C_2 R_4 \right) \right]}{\left( R_1 + R_4 + R_1 \left( \frac{C_2}{C_3} \right) \right)^2 + \left( \frac{-1}{\omega C_3} + \omega R_1 C_2 R_4 \right)^2},$$

and finally:

$$V_{out} = \text{Re}(\mathbf{V}_{DE}) = \frac{V_0 R_4 \left( R_1 + R_4 + R_1 \left( \frac{C_2}{C_3} \right) \right) \cos(\omega t) + V_0 R_4 \left( \frac{-1}{\omega C_3} + \omega R_1 C_2 R_4 \right) \sin(\omega t)}{\left( R_1 + R_4 + R_1 \left( \frac{C_2}{C_3} \right) \right)^2 + \left( \frac{-1}{\omega C_3} + \omega R_1 C_2 R_4 \right)^2}$$

Note that this result has the general form  $V_{out} = V_0 R_4 \frac{A \cos(\omega t) + B \sin(\omega t)}{A^2 + B^2}$ , where

$A = R_1 + R_4 + R_1 \frac{C_2}{C_3}$  and  $B = \frac{-1}{\omega C_3} + \omega R_1 C_2 R_4$ . The amplitude of the output will then be:

$$V_{out}^{\max} = \frac{V_0 R_4}{\sqrt{A^2 + B^2}},$$

and the transfer function T will then be given by:

$$T = \frac{V_{out}^{rms}}{V_{in}^{rms}} = \frac{V_{out}^{peak}}{V_{in}^{peak}} = \frac{R_4}{\sqrt{A^2 + B^2}} = \frac{R_4}{\sqrt{\left( R_1 + R_4 + R_1 \frac{C_2}{C_3} \right)^2 + \left( \omega R_1 C_2 R_4 - \frac{1}{\omega C_3} \right)^2}}$$